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1984 J. Phys. A: Math. Gen. 17 L961

(http://iopscience.iop.org/0305-4470/17/18/005)

J. Phys. A: Math. Gen. 17 (1984) L961-L964. Printed in Great Britian

LETTER TO THE EDITOR

Finite-size scaling in strips: Antiperiodic boundary conditions

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Received 3 October 1984

Abstract. The exact values of the universal amplitude A relating the correlation length $\xi \sim L/A$ to the strip width L at the critical point of Ising and 3-state Potts models are obtained for the case of antiperiodic, or twisted, boundary conditions. Predictions are also made for the finite-size scaling behaviour of the interfacial tension.

For an infinitely long two-dimensional strip of finite width L, the theory of finite-size scaling (Barber 1983) predicts that the inverse correlation length ξ^{-1} at the critical point of the infinite system should tend to zero like A/L, where the amplitude A is universal, but may depend on the boundary conditions. In a previous letter (Cardy 1984a), we allowed that A is equal to $\pi\eta$ and $\frac{1}{2}\pi\eta_{\|}$ for periodic and free boundary conditions respectively, for any isotropic model. The result for the periodic case has been confirmed by exact and numerical calculations in many models (Luck 1982, Derrida and de Seze 1982, Nightingale and Blöte 1983, Privman and Fisher 1984).

In this letter we calculate A for the case of antiperiodic or twisted boundary conditions for two examples where such a boundary condition is appropriate, namely the Ising (Z_2) and 3-state Potts (Z_3) models. First, consider an Ising model defined on the strip $0 \leqslant x \leqslant L$, $-\infty \leqslant y \leqslant +\infty$ with antiperiodic boundary conditions $\sigma(x+L, y) = -\sigma(x, y)$, where $\sigma(x, y)$ is the local value of the spin. This problem may be transformed into one with periodic boundary conditions by inserting an antiferromagnetic seam, that is by changing the sign of the exchange interaction on those bonds which intersect an arbitrarily chosen path which begins at $y = -\infty$ and ends at $y = +\infty$. This is equivalent to inserting two disorder operators μ (dual to σ) at $y = \pm\infty$ (Kadanoff and Ceva 1971). Explicitly,

$$\langle \sigma(x_1, y_1)\sigma(x_2, y_2)\rangle_A = \lim_{\substack{y_1' \to -\infty \\ y_2' \to +\infty}} \frac{\langle \mu(x_1', y_1')\sigma(x_1, y_1)\sigma(x_2, y_2)\mu(x_2', y_2')\rangle}{\langle \mu(x_1', y_1')\mu(x_2', y_2')\rangle} \tag{1}$$

where the correlation functions on the right-hand side are evaluated in the strip with periodic boundary conditions. For $y_2 - y_1 \to \infty$ this quantity should decay like $\exp[-(y_2 - y_1)/\xi]$.

The correlation functions in (1) may also be written in an operator notation, introducing the transfer matrix for the strip $\hat{T} = e^{-\hat{H}a}$. $\hat{H}$ is the infinitesimal y-translation operator in the continuum limit, as the lattice spacing $a \to 0$. Introducing the eigenvalues E_n and corresponding eigenstates $|n\rangle$ of $\hat{H}$, (1) can be rewritten for $|y_2 - y_1| \gg L$ as

$$\frac{\langle 0|\mu|1\rangle \exp[-E_1(y_1 - y_1')]\langle 1|\sigma|2\rangle \exp[-E_2(y_2 - y_1)]\langle 2|\sigma|1\rangle \exp[-E_1(y_2' - y_2)]\langle 1|\mu|0\rangle}{\langle 0|\mu|1\rangle \exp[-E_1(y_2' - y_1')]\langle 1|\mu|0\rangle} \tag{2}$$

where E_1 and E_2 are the lowest eigenvalues of $\hat{H}$ such that the matrix elements $\langle 0|\mu|1\rangle$ and $\langle 1|\sigma|2\rangle$ are non-zero. In writing (2) we have normalised $\hat{H}$ so that $E_0=0$, and suppressed the x dependence. From this expression we see that $\xi^{-1}=E_2-E_1$. Now E_1 gives the correlation length of the $\langle\mu\mu\rangle$ correlation function in the strip, which is the same as that of $\langle\sigma\sigma\rangle$, by duality. Thus $E_1=\pi\eta/L=\pi/4L$ (Cardy 1984a). To evaluate E_2 we may consider the same correlation function as appears in the numerator of (1), but in the limit $y_2-y_1\to\infty$ with $y_1'-y_1$ and $y_2'-y_2$ fixed, and assume that the same intermediate state $|2\rangle$ will dominate. In this limit, we can use the short-distance expansion (Kadanoff and Ceva 1971) $\sigma\mu\sim\psi$ (where ψ is the Ising fermion) to argue that the dependence on y_2-y_1 is the same as that of the correlation function

$$\langle\psi(x_1, y_1)\psi(x_2, y_2)\rangle. \tag{3}$$

In the bulk, it is known (Dotsenko 1984) that this correlation function behaves like $(z_1-z_2)^{-1}$, where $z_j=x_j+\mathrm{i}y_j$. Note that this changes sign when z_1 and z_2 are interchanged, consistent with the fermionic interpretation of ψ. The conformal invariance methods (Cardy 1984a) then imply that in the strip this correlation function decays with an inverse correlation length $E_2=\pi/L$. Thus in this case $\xi^{-1}=3\pi/4L$.

An analogous argument holds for the Z_3 model, for which σ takes the values 1, $\exp(2\pi\mathrm{i}/3)$, $\exp(-2\pi\mathrm{i}/3)$, and we impose twisted boundary conditions $\sigma(x+L, y)=\exp(2\pi\mathrm{i}/3)\sigma(x, y)$. In that case the short-distance product of an order and a disorder operator gives a parafermion ψ (Fradkin and Kadanoff 1980) whose spin is $\frac{1}{3}$. This means that the bulk correlation function $\langle\psi(z_1)\psi(z_2)\rangle$ behaves like $(z_1-z_2)^{-2\Delta}(\bar{z}_1-\bar{z}_2)^{-2\bar{\Delta}}$ where $\Delta-\bar{\Delta}=\frac{1}{3}$. In the Z_3 model, the values of Δ and $\bar{\Delta}$ allowed by conformal invariance are (Dotsenko 1984, Friedan *et al* 1984) 0, $\frac{2}{5}$, $\frac{7}{5}$, 3, $\frac{1}{15}$, $\frac{2}{3}$. We conclude that $\Delta=\frac{2}{5}$, $\bar{\Delta}=\frac{1}{15}$, so that ψ has scaling dimension $x=\Delta+\bar{\Delta}=\frac{7}{15}$. For this case, then, $E_2=14\pi/15L$ and $E_1=4\pi/15L$, so that $\xi^{-1}=2\pi/3L$.

The complete set of predictions for A/π for the three kinds of boundary conditions are shown in table 1. We now compare these results with existing numerical data. Burkhardt and Guim (1984a, b) have studied isotropic Ising models in strips with periodic and free boundary conditions, and the quantum Ising chain with all three types of boundary conditions. Our results agree with theirs. Note that for the quantum model the amplitude A itself is not universal (see Penson and Kolb 1984 and Nightingale and Blöte 1984); but ratios of amplitudes with different boundary conditions should be.

Gehlen *et al* (1984) have considered anisotropic quantum Ising models and quantum Z_3 models. For Ising models they found that A was independent of the boundary conditions in clear contradiction to both our results and those of Burkhardt and Guim. For the Z_3 model they found amplitudes approximately in the ratios $\frac{2}{5}:1:1$ for periodic,free and twisted boundary conditions respectively, in agreement with our exact results.

Table 1. Values of A/π for Ising (Z_2) and 3-state Potts (Z_3) models. The values of $\eta_\parallel$ used in the second row come from Cardy (1984b) and McCoy and Wu (1967) for the Z_2 case.

	Z_2	Z_3
Periodic	$\frac{1}{4}$	$\frac{4}{15}$
Free	$\frac{1}{2}$	$\frac{2}{3}$
Twisted	$\frac{3}{4}$	$\frac{2}{3}$

Duality arguments may also be used to predict the beahviour of the interfacial tension Σ at the critical point in a finite system. This quantity may be defined in terms of the difference between the free energies per unit length of strips with periodic and antiperiodic boundary conditions. For the infinite system, Σ is supposed to vanish like $(T-T_c)^\nu$ in two dimensions (Widom 1972). Standard finite-size scaling arguments then imply that $\Sigma \sim B/L$ for a strip of width L at T_c, with B a universal amplitude. Σ may also be calculated in terms of correlation functions of disorder variables

$$\Sigma = -\lim_{y_2-y_1\to\infty} (y_2-y_1)^{-1} \ln\langle \mu(x_1, y_1)\mu(x_2, y_2)\rangle. \tag{3}$$

However, this correlation function is dual to $\langle\sigma\sigma\rangle$, whose behaviour we know (Cardy 1984a). We conclude that $B=\pi\eta$ for this definition of Σ.

An alternative definition of Σ is given by the difference in free energies for the two cases of fixed boundary conditions: (a) when the spins on either side are fixed in the same state; (b) when they are fixed in different states. In that case (3) still holds, but $\langle\mu\mu\rangle$ is now dual to the correlation function $\langle\sigma\sigma\rangle$ evaluated with *free* boundary conditions. In that case, then, $B=\frac{1}{2}\pi\eta_\parallel$. This definition of Σ is natural for the q-state Potts model. For the percolation limit ($q\to 1$) it may be shown that the probability that the two sides of a strip of width L and length L' are connected by at least one path is

$$P = 1-\exp(-L'\Sigma) \tag{4}$$

in the limit $L'\to\infty$, with the above definition of Σ. With the exact value $\eta_\parallel=\frac{2}{3}$ (Cardy 1984b) we therefore predict that

$$P \sim 1-\exp(-\pi L'/3L) \tag{5}$$

in the limits $L\to\infty$, $L'/L\to\infty$, at the percolation threshold p_c.

This work was supported by the National Science Foundation under Grant No PHY83-13324. The author is extremely grateful for the hospitality offered by the Service de Physique Théorique, CEN-Saclay, where this work was initiated, to T Burkhardt for communicating his resutls prior to publication, and to B Derrida for useful discussions.

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